

COHOMOLOGY OF TWISTED \mathcal{D} -MODULES ON \mathbb{P}^1 OBTAINED AS EXTENSIONS FROM \mathbb{C}^\times

CLAUDE EICHER

ABSTRACT. We construct twisted \mathcal{D} -modules on the projective line \mathbb{P}^1 that are equivariant for the action of the diagonal torus subgroup of SL_2 . In the most interesting case these arise as extensions from local systems on \mathbb{C}^\times . We discuss their subquotient structure. Their sheaf cohomology groups are weight modules for the Lie algebra \mathfrak{sl}_2 . We also discuss their subquotient structure and in case these modules are not the familiar highest or lowest weight modules, we give an explicit presentation for them. Our computations illustrate some basic \mathcal{D} -module concepts and the Beilinson-Bernstein equivalence. They are the first step in a program that aims to describe categories of modules over semisimple and affine Kac-Moody Lie algebras that are next to highest (or lowest) weight via \mathcal{D} -modules on the flag variety.

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1. INTRODUCTION AND ORGANIZATION

Consider a local system on \mathbb{C}^\times given by the monodromy $e^{2\pi i\alpha}$, where $\alpha \in \mathbb{C}$ is a parameter. It can be described by the differential equation $(x\partial_x - \alpha)f(x) = 0$ and hence as a connection on \mathbb{C}^\times . The formulation of these objects in the language of \mathcal{D} -modules is given in § 2.5. The theory of \mathcal{D} -modules provides extensions (direct images) of such objects, living on \mathbb{C}^\times , to \mathcal{D} -modules on the projective line \mathbb{P}^1 . The simplest such extension is the mere sheaf-direct image, while the $!$ -extension is constructed from it via a duality functor (§ 2.2). In addition, with the aim of being systematic, we introduce yet another extension denoted $j_{!x,z}$ in § 2.6. We analyze all these extensions in § 2.6 by determining their subquotients (composition series) in the category of \mathcal{D} -modules. In addition to \mathbb{C}^\times we have also included for comparison in § 2.4 the easier and even better known case of the affine line. As a technical aside, let us mention that we work with right \mathcal{D} -modules starting from § 2.4 because the formation of direct images is easier than for left \mathcal{D} -modules.

In § 4 we compute the sheaf cohomology groups of our extensions as \mathfrak{sl}_2 -modules using the Čech complex for the standard affine open cover of \mathbb{P}^1 . They turn out to be weight modules (§ 3) for \mathfrak{sl}_2 with trivial central character and lowest or highest weight modules in case of the affine line. The identification of these cohomology groups can be understood as a building block in the description of the category of \mathfrak{sl}_2 -weight modules with trivial central character via \mathcal{D} -modules on \mathbb{P}^1 . The basic reason for the appearance of the subspace \mathbb{C}^\times is that SL_2 acts on \mathbb{P}^1 and $\mathbb{C}^\times \sqcup \{0\} \sqcup \{\infty\}$ is the stratification of \mathbb{P}^1 by orbits of the diagonal torus subgroup of SL_2 . Similarly, the affine lines $\mathbb{P}^1 \setminus \{\infty\}$ resp. $\mathbb{P}^1 \setminus \{0\}$ are the open orbit of the Borel resp. opposite Borel subgroup of SL_2 . The corresponding \mathcal{D} -module extensions (the $!$ - resp. $*$ -extension is usually called standard resp. costandard object in this context) acquire an equivariance property w.r.t. the subgroup. The equivariance is addressed in § 2.7 and in Remark 4.4. Further, we discuss in § 4.1 how the change of variables $x \mapsto x^{-1}$ induces auto-equivalences $(\cdot)^-$ of the category of \mathcal{D} - resp. \mathfrak{sl}_2 -modules.

In § 5 we review the notion of twisted \mathcal{D} -modules. In our case the twist parameter is an arbitrary integer λ . Subsequently we determine the sheaf cohomology groups of our extensions considered now as twisted \mathcal{D} -modules, thereby generalizing the results of § 4. They are again weight modules for \mathfrak{sl}_2 with central character depending on λ . In § 5.2.2 we analyze the structure (composition series) of a weight module (depending on

the parameters α and λ), which appears as global sections of our extensions. The main results of this work are gathered into Theorems 5.1, 5.2, 5.3, 5.4 and 5.5. These are all proven by direct computations with the Čech complex. While parts of these results have previously appeared in the literature, our treatment considers all choices of the parameter α and all integral twists λ for \mathcal{D} -modules. As explained in Remarks 4.4 and 5.7, the results illustrate the Beilinson-Bernstein equivalence.

Finally, we sketch in § 6 how this work fits into a larger program by replacing SL_2 by any semisimple algebraic group over \mathbb{C} or the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_2$.

NOTATION

We use the symbol § when referring to entire sections. When making a statement that holds for multiple indices separately, for brevity we simply list the indices, see e.g. § 2.3: we write $\mathbb{C}_{x,z}$ instead of $\mathbb{C}_x, \mathbb{C}_z$. We denote by curly letters sheaves \mathcal{F} and their sections over an open U by $\mathcal{F}(U)$. If G is a linear algebraic group over \mathbb{C} , then $\mathrm{Lie} G$ denotes its Lie algebra. All varieties we consider are algebraic varieties over \mathbb{C} . By a local system, we mean a vector bundle with flat connection. We have tried to isolate statements that do not depend strongly on the rest of the text or are used multiple times into remarks.

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2. UNTWISTED \mathcal{D} -MODULES

2.1. Left and right \mathcal{D} -modules. We recall a few generalities on left and right \mathcal{D} -modules, see e.g. [Kas00][section 1.4]. Let X be a smooth variety. Let \mathcal{D}_X be the sheaf of algebraic differential operators on X . We denote by $\mathcal{D}_X \text{ mod}$ resp. $\text{mod } \mathcal{D}_X$ the category of left resp. right \mathcal{D}_X -modules quasicoherent as \mathcal{O}_X -modules. Then the structure sheaf \mathcal{O}_X and the sheaf of volume forms Ω_X is naturally a left resp. right \mathcal{D}_X -module. Further we have an isomorphism of sheaves of \mathbb{C} -algebras

$$\mathcal{D}_X^{\mathrm{op}} \xrightarrow{\cong} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}. \quad (2.1)$$

In (étale) local coordinates $(x_j)_{1 \leq j \leq n}$ on X it is given by $p \mapsto dx \otimes^t p \otimes dx^{\otimes -1}$. Here $dx = dx_1 \wedge \cdots \wedge dx_n$ and $dx^{\otimes -1}$ satisfies $\langle dx, dx^{\otimes -1} \rangle = 1$. The *formal adjoint* ${}^t(\cdot)$ is the unique anti-involution of \mathcal{D}_X satisfying ${}^t(x_j) = x_j$ and ${}^t(\partial_{x_j}) = -\partial_{x_j}$. Via the isomorphism (2.1) it becomes manifest that Ω_X is a left $\mathcal{D}_X^{\mathrm{op}}$ -module and that a left $\mathcal{D}_X^{\mathrm{op}}$ -module is the same as a Ω_X -twisted left \mathcal{D}_X -module (the notion of a twisted \mathcal{D} -module will be recalled in § 5.1 below). The functor $(\cdot)^r : \mathcal{D}_X \text{ mod} \rightarrow \text{mod } \mathcal{D}_X$ given on objects by $\mathcal{M} \mapsto \mathcal{M}^r = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is an exact equivalence. Similarly, the functor

$(\cdot)^l : \text{mod } \mathcal{D}_X \rightarrow \mathcal{D}_X \text{ mod}, \mathcal{M} \mapsto \mathcal{M}^l = \Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} \mathcal{M}$, is an exact equivalence inverse to $(\cdot)^r$.

2.2. Holonomic duality \mathbb{D} . For the notions of this subsection, see e.g. [Kas00][section 4.11]. Let us assume $\dim X = 1$. Let \mathcal{M} be a holonomic right \mathcal{D}_X -module (in particular \mathcal{M} is required to be coherent). Its dual is defined as $\mathbb{D}\mathcal{M} = \mathcal{E}xt_{\mathcal{D}_X}^1(\mathcal{M}, \mathcal{D}_X)^r$. Here $\mathcal{E}xt_{\mathcal{D}_X}^1(\mathcal{M}, \mathcal{D}_X)$ is naturally a left \mathcal{D}_X -module, to which we apply $(\cdot)^r$. \mathbb{D} defines an exact contravariant auto-equivalence of the category of holonomic \mathcal{D}_X -modules.

Remark 2.1. We recall a formula for $\mathbb{D}\mathcal{M}$ in case of $X = \mathbb{C}$, the affine line, and $\mathcal{M} = \mathcal{D}_{\mathbb{C}}/p\mathcal{D}_{\mathbb{C}}$, where p is a nonzero polynomial in x and ∂_x . Consider the free resolution of right $\mathcal{D}_{\mathbb{C}}$ -modules

$$0 \rightarrow \mathcal{D}_{\mathbb{C}} \xrightarrow{p} \mathcal{D}_{\mathbb{C}} \rightarrow \mathcal{M} \rightarrow 0. \quad (2.2)$$

The map $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{D}_X) \xrightarrow{\cong} \mathcal{D}_X, \phi \mapsto \phi(1)$, is an isomorphism of left \mathcal{D}_X -modules. We apply $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\cdot, \mathcal{D}_{\mathbb{C}})$ to (2.2) and get an exact sequence of left $\mathcal{D}_{\mathbb{C}}$ -modules

$$0 \leftarrow \mathcal{D}_{\mathbb{C}} \xleftarrow{p} \mathcal{D}_{\mathbb{C}} \leftarrow \mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{M}, \mathcal{D}_{\mathbb{C}}) \leftarrow 0.$$

It follows $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{M}, \mathcal{D}_{\mathbb{C}}) = 0$ and $\mathcal{E}xt_{\mathcal{D}_{\mathbb{C}}}^1(\mathcal{M}, \mathcal{D}_{\mathbb{C}}) \cong \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}p$. With (2.1) we conclude that $\Omega_{\mathbb{C}} \otimes_{\mathcal{O}_{\mathbb{C}}} \mathcal{D}_{\mathbb{C}} \rightarrow \mathcal{D}_{\mathbb{C}} \otimes_{\mathcal{O}_{\mathbb{C}}} \Omega_{\mathbb{C}}, dx \otimes p \mapsto {}^t p \otimes dx$, is an isomorphism of left $\mathcal{D}_{\mathbb{C}}^{\text{op}}$ -modules and hence

$$\mathbb{D}\mathcal{M} = \Omega_{\mathbb{C}} \otimes_{\mathcal{O}_{\mathbb{C}}} (\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}p) = \mathcal{D}_{\mathbb{C}} \otimes_{\mathcal{O}_{\mathbb{C}}} \Omega_{\mathbb{C}} / ({}^t p) \mathcal{D}_{\mathbb{C}} \otimes_{\mathcal{O}_{\mathbb{C}}} \Omega_{\mathbb{C}} \cong \mathcal{D}_{\mathbb{C}} / {}^t p \mathcal{D}_{\mathbb{C}}$$

is an isomorphism of right $\mathcal{D}_{\mathbb{C}}$ -modules. This formula for $\mathbb{D}\mathcal{M}$ will be applied later.

2.3. Geometric setup. In the remaining part of the text we will be concerned with the following geometric setup. Let x be a global coordinate on \mathbb{P}^1 considered as a variety. Then $z = x^{-1}$ is another global coordinate. Consider the open subsets $\mathbb{C}_x = (x \neq \infty)$ and $\mathbb{C}_z = (z \neq \infty)$ of \mathbb{P}^1 and the corresponding open embeddings $j_{x,z} : \mathbb{C}_{x,z} \hookrightarrow \mathbb{P}^1$ and $j : \mathbb{C}^{\times} = \mathbb{C}_x \cap \mathbb{C}_z \hookrightarrow \mathbb{P}^1$. Thus, we have a commutative diagram of open embeddings

$$\begin{array}{ccc} & \mathbb{C}_x & \\ \nearrow & & \searrow \\ \mathbb{C}^{\times} & & \mathbb{P}^1 \\ \searrow & & \nearrow \\ & \mathbb{C}_y & \end{array}.$$

The closed embeddings of the complements will be denoted by $\iota_{x,z}$. We will also work with homogeneous coordinates $[z_0 : z_1]$ on \mathbb{P}^1 related to the coordinate x by $x = z_1/z_0$.

2.4. Direct images of $\Omega_{\mathbb{C}_{x,z}}$. *In the subsequent text we will only consider right \mathcal{D} -modules unless stated otherwise.* First recall that the action of ∂_x on $\Omega_{\mathbb{C}_x}(\mathbb{C}_x) = \mathbb{C}[x]dx$ is given by $(x^n dx)\partial_x = -nx^{n-1}dx$. Also recall that the sheaf direct image $j_{x*}\Omega_{\mathbb{C}_x}$ of $\Omega_{\mathbb{C}_x}$ is naturally a $\mathcal{D}_{\mathbb{P}^1}$ -module. Let us describe it on the open affine cover $\{\mathbb{C}_x, \mathbb{C}_z\}$ of \mathbb{P}^1 . We have $(j_{x*}\Omega_{\mathbb{C}_x})|_{\mathbb{C}_z} \cong \mathcal{D}_{\mathbb{C}_z}/z\partial_z\mathcal{D}_{\mathbb{C}_z}$ as

$$(j_{x*}\Omega_{\mathbb{C}_x})(\mathbb{C}_z) = \Omega_{\mathbb{C}_x}(\mathbb{C}_z \cap \mathbb{C}_x) = \mathbb{C}[x, x^{-1}]dx = \mathbb{C}[z, z^{-1}]dz \xrightarrow{\cong} (\mathcal{D}_{\mathbb{C}_z}/z\partial_z\mathcal{D}_{\mathbb{C}_z})(\mathbb{C}_z).$$

The last isomorphism is given by $\frac{dz}{z} \mapsto \bar{1}$. Here and in the following \bar{a} will denote the class of an element a in a quotient. We have the exact sequence

$$0 \rightarrow \Omega_{\mathbb{C}_z} = \mathcal{D}_{\mathbb{C}_z}/\partial_z\mathcal{D}_{\mathbb{C}_z} \rightarrow (j_{x*}\Omega_{\mathbb{C}_x})|_{\mathbb{C}_z} = \mathcal{D}_{\mathbb{C}_z}/z\partial_z\mathcal{D}_{\mathbb{C}_z} \rightarrow \mathcal{D}_{\mathbb{C}_z}/z\mathcal{D}_{\mathbb{C}_z} \rightarrow 0 \quad (2.3)$$

of $\mathcal{D}_{\mathbb{C}_z}$ -modules. The first map sends $\bar{1} \mapsto \bar{z}$. Trivially, $(j_{x*}\Omega_{\mathbb{C}_x})|_{\mathbb{C}_x} = \Omega_{\mathbb{C}_x}$. Similarly, $(j_{z*}\Omega_{\mathbb{C}_z})|_{\mathbb{C}_x} \cong \mathcal{D}_{\mathbb{C}_x}/x\partial_x\mathcal{D}_{\mathbb{C}_x}$.

Remark 2.2. Let us recall the exact triangle from a pair of complementary closed and open embedding. Let $\iota : Z \hookrightarrow X$ be a closed immersion of a smooth variety Z into a smooth variety X . Let $j : U = X \setminus Z \hookrightarrow X$ be the open embedding of the complement. If \mathcal{M} is an object of the bounded derived category of \mathcal{D}_X -modules we have an exact triangle [Ber][p. 8]

$$\iota_*\iota^!\mathcal{M} \rightarrow \mathcal{M} \rightarrow j_*j^{-1}\mathcal{M} \xrightarrow{[1]}. \quad (2.4)$$

We comment on the direct and inverse images involved: Recall that j^{-1} , the restriction to the open U , is exact. $Rj_* \equiv j_*$ is the right derived direct image of sheaves. In (2.4) $\mathcal{M} \rightarrow j_*j^{-1}\mathcal{M}$ is the unit of the adjunction. According to [Ber][p. 5] ι_* is exact and $R\iota^! \equiv \iota^!$ is right derived since ι is a closed embedding. ι_* is left adjoint to $\iota^!$ and in (2.4) $\iota_*\iota^!\mathcal{M} \rightarrow \mathcal{M}$ is the counit of the adjunction. $\iota_*\iota^! = R\Gamma_Z$ is the right derived functor of sections supported in Z . Also recall from [Kas00][section 3.4] that if $\mathcal{M} = \Omega_X$ (placed in degree zero) and Z is a complete intersection of codimension d in X then $H^l(R\Gamma_Z(\Omega_X)) = 0$ for $l \neq d$ and $\mathcal{B}_{Z|X} = H^d(R\Gamma_Z(\Omega_X))$ is the \mathcal{D} -module of distributions on X with singularities along Z .

Now we consider the above remark for $\iota = \iota_x$, $j = j_x$ and $\mathcal{M} = \Omega_{\mathbb{P}^1}$ (in degree zero). We find $\iota_x^!\Omega_{\mathbb{P}^1} = \mathbb{C}[-1]$. (Clearly $\Omega_{\mathbb{P}^1}$ does not have sections supported at $(x = \infty)$.) The long exact sequence associated to (2.4) is

$$0 \rightarrow \Omega_{\mathbb{P}^1} \rightarrow j_{x*}\Omega_{\mathbb{C}_x} \rightarrow \iota_{x*}\mathbb{C} \rightarrow 0 \rightarrow \dots \quad (2.5)$$

The restriction of this sequence to the open \mathbb{C}_z is (2.3). Here $\iota_{x*}\mathbb{C} = H^1(R\Gamma_{(x=\infty)}(\Omega_{\mathbb{P}^1})) = \mathcal{B}_{(x=\infty)|\mathbb{P}^1}$ is the skyscraper \mathcal{D} -module supported at $x = \infty$.

2.4.1. Subquotients of $j_{x*}\Omega_{\mathbb{C}_x}$. These are clear from (2.5) and the fact that $\Omega_{\mathbb{P}^1}$ and $\iota_{x*}\mathbb{C}$ are simple $\mathcal{D}_{\mathbb{P}^1}$ -modules.

2.4.2. *Subquotients of $j_{x!}\Omega_{\mathbb{C}_x}$.* The direct image $j_{x!}$ is defined as $j_{x!} = \mathbb{D}j_x\mathbb{D}$, see e.g. [Ber][p. 18]. Applying \mathbb{D} to (2.5) we find with $\mathbb{D}\Omega_{\mathbb{P}^1} \cong \Omega_{\mathbb{P}^1}$ and $\mathbb{D}\iota_{x*}\mathbb{C} \cong \iota_{x*}\mathbb{C}$ the exact sequence

$$0 \leftarrow \Omega_{\mathbb{P}^1} \leftarrow j_{x!}\Omega_{\mathbb{C}_x} \leftarrow \iota_{x*}\mathbb{C} \leftarrow 0. \quad (2.6)$$

According to (2.3) and Remark 2.1 this sequence restricts to

$$0 \leftarrow \Omega_{\mathbb{C}_z} = \mathcal{D}_{\mathbb{C}_z}/\partial_z\mathcal{D}_{\mathbb{C}_z} \leftarrow (j_{x!}\Omega_{\mathbb{C}_x})|_{\mathbb{C}_z} = \mathcal{D}_{\mathbb{C}_z}/\partial_z z\mathcal{D}_{\mathbb{C}_z} \leftarrow \mathcal{D}_{\mathbb{C}_z}/z\mathcal{D}_{\mathbb{C}_z} \leftarrow 0 \quad (2.7)$$

on \mathbb{C}_z . The first map (from the right) sends $\bar{1} \mapsto \bar{\partial}_z$.

2.5. Definition of $\Omega_{\mathbb{C}^\times}^{(\alpha)}$. For $\alpha \in \mathbb{C}$ define $\Omega_{\mathbb{C}^\times}^{(\alpha)} = \mathcal{D}_{\mathbb{C}^\times}/(x\partial_x - \alpha)\mathcal{D}_{\mathbb{C}^\times}$. This definition depends on the choice of the global coordinate x . In the global coordinate $z = x^{-1}$ we find $\Omega_{\mathbb{C}^\times}^{(\alpha)} = \mathcal{D}_{\mathbb{C}^\times}/(z\partial_z + \alpha)\mathcal{D}_{\mathbb{C}^\times}$. As a special case we have

$$\Omega_{\mathbb{C}^\times}^{(-1)} = \mathcal{D}_{\mathbb{C}^\times}/\partial_x x\mathcal{D}_{\mathbb{C}^\times} = \mathcal{D}_{\mathbb{C}^\times}/\partial_x\mathcal{D}_{\mathbb{C}^\times} = \Omega_{\mathbb{C}^\times}.$$

Remark 2.3. We list elementary properties of $\Omega_{\mathbb{C}^\times}^{(\alpha)}$.

- (1) $\Omega_{\mathbb{C}^\times}^{(\alpha)}$ is a free $\mathcal{O}_{\mathbb{C}^\times}$ -module of rank one. Indeed, $\mathcal{D}_{\mathbb{C}^\times}(\mathbb{C}^\times)/(x\partial_x - \alpha)\mathcal{D}_{\mathbb{C}^\times}(\mathbb{C}^\times) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\bar{1}x^n$. Since $(\bar{1}x^n)x\partial_x = (\alpha - n)\bar{1}x^n$ our module is a direct sum of one dimensional eigenspaces of $x\partial_x$ with eigenvalues $\alpha + \mathbb{Z}$.
- (2) $\Omega_{\mathbb{C}^\times}^{(\alpha)} \cong \Omega_{\mathbb{C}^\times}^{(\beta)}$ if and only if $\alpha - \beta \in \mathbb{Z}$. From the eigenspace decomposition for $x\partial_x$ described in (1) we find that $\alpha - \beta \notin \mathbb{Z}$ implies $\Omega_{\mathbb{C}^\times}^{(\alpha)} \not\cong \Omega_{\mathbb{C}^\times}^{(\beta)}$. On the other hand, for $n \in \mathbb{Z}$ $\phi : \Omega_{\mathbb{C}^\times}^{(\alpha)} \rightarrow \Omega_{\mathbb{C}^\times}^{(\alpha+n)}$, $\bar{1} \mapsto \bar{1}x^n$, extends uniquely to a morphism of $\mathcal{D}_{\mathbb{C}^\times}$ -modules because of $\phi(\bar{1}x\partial_x) = \phi(\bar{1})x\partial_x = \alpha\phi(\bar{1})$. Further, ϕ is invertible.
- (3) $\mathbb{D}\Omega_{\mathbb{C}^\times}^{(\alpha)} \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^\times}}(\Omega_{\mathbb{C}^\times}^{(\alpha)}, \Omega_{\mathbb{C}^\times})^r \cong \Omega_{\mathbb{C}^\times}^{(-\alpha)}$. The first isomorphism is known to hold, as consequence of (1), see e.g. [Kas00][Lemma 3.13]. We now explain the second isomorphism. Recall that $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^\times}}(\Omega_{\mathbb{C}^\times}^{(\alpha)}, \Omega_{\mathbb{C}^\times})$ is a left $\mathcal{D}_{\mathbb{C}^\times}$ -module if we let the vector field $\xi \in \mathcal{D}_{\mathbb{C}^\times}$ act on a local section ψ by

$$(\xi\psi)(m) = -\psi(m)\xi + \psi(m\xi), \quad (2.8)$$

see e.g. [ea87][Chapter VI, 3.4]. This implies that the map between the right $\mathcal{D}_{\mathbb{C}^\times}$ -modules

$$\Omega_{\mathbb{C}^\times} \otimes_{\mathcal{O}_{\mathbb{C}^\times}} \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^\times}}(\Omega_{\mathbb{C}^\times}^{(\alpha)}, \Omega_{\mathbb{C}^\times}) \rightarrow \Omega_{\mathbb{C}^\times}, \quad dx \otimes \psi \mapsto \psi(\bar{1}),$$

which is an isomorphism of $\mathcal{O}_{\mathbb{C}^\times}$ -modules, sends

$$(dx \otimes \psi)x\partial_x \mapsto \psi(\bar{1})(-1 - \alpha + x\partial_x),$$

where we applied (2.8). Hence, the global sections of $\mathcal{H}om_{\mathcal{O}_{\mathbb{C}^\times}}(\Omega_{\mathbb{C}^\times}^{(\alpha)}, \Omega_{\mathbb{C}^\times})^r$ decompose into a direct sum of one dimensional eigenspaces for the eigenvalues $-\alpha + \mathbb{Z}$ for the action of $x\partial_x$. Hence, by (2), the second isomorphism. Also note that $\mathbb{D}\Omega_{\mathbb{C}^\times}^{(\alpha)} \cong \Omega_{\mathbb{C}^\times}^{(-\alpha)}$ follows directly from remark 2.1 and (2) as ${}^t(x\partial_x - \alpha) = -x\partial_x - \alpha - 1$.

2.6. Direct images of $\Omega_{\mathbb{C}^\times}^{(\alpha)}$. Let us describe the direct images j_* and $j_!$ of $\Omega_{\mathbb{C}^\times}^{(\alpha)}$. In addition, we will consider a functor $j_{!x,z} : \text{mod } \mathcal{D}_{\mathbb{C}^\times} \rightarrow \text{mod } \mathcal{D}_{\mathbb{P}^1}$ defined on objects as follows. Let \mathcal{M} be a $\mathcal{D}_{\mathbb{C}^\times}$ -module. We glue the $\mathcal{D}_{\mathbb{C}_x}$ -module $j_x^{-1}j_!\mathcal{M}$ and the $\mathcal{D}_{\mathbb{C}_z}$ -module $j_z^{-1}j_*\mathcal{M}$ along \mathbb{C}^\times to a $\mathcal{D}_{\mathbb{P}^1}$ -module $j_{!x,z}\mathcal{M}$. Thus, informally $j_{!x,z}$ can be described as the $!$ -extension at $x = 0$ and the $*$ -extension at $z = 0$. Of course we have a similar functor $j_{*x!z}$ satisfying $j_{*x!z} = \mathbb{D} j_{!x,z} \mathbb{D}$.

Lemma 2.1. Let $\alpha \notin \mathbb{Z}$.

- (1) $j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$ is simple. The restriction to \mathbb{C}_x is

$$j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_x} \cong \mathcal{D}_{\mathbb{C}_x}/(x\partial_x - \alpha)\mathcal{D}_{\mathbb{C}_x}. \quad (2.9)$$

The canonical map $j_!\Omega_{\mathbb{C}^\times}^{(\alpha)} \rightarrow j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$, see e.g. [Ber][p. 18], is an isomorphism.

- (2) $j_{!x,z}\Omega_{\mathbb{C}^\times}^{(\alpha)} \cong j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$

Proof. (1). We have the decomposition

$$j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}_x) = \bigoplus_{n \in \mathbb{Z}} \left(j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}_x) \right)_{\alpha+n} = \bigoplus_{n \in \mathbb{Z}} \left(\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}^\times) \right)_{\alpha+n}$$

into one dimensional eigenspaces for $x\partial_x$ and x and ∂_x intertwine those:

$$\left(\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}^\times) \right)_{\alpha+n+1} \xrightleftharpoons[\cdot\partial_x]{\cdot x} \left(\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}^\times) \right)_{\alpha+n}.$$

As

$$\cdot x\partial_x = (\alpha + n + 1) \text{id}_{\left(\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}^\times) \right)_{\alpha+n+1}} \quad \cdot \partial_x x = (\alpha + n + 1) \text{id}_{\left(\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}^\times) \right)_{\alpha+n}}$$

the maps x and ∂_x are invertible for each n if $\alpha \notin \mathbb{Z}$. Let \mathcal{M} be a submodule of $j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$. Let $s \in \mathcal{M}(\mathbb{C}_x)$. Writing $s = \sum_{j=1}^n s_j$ with $s_j x\partial_x = \lambda_j s_j$ and distinct λ_j we conclude $s_j \in \mathcal{M}(\mathbb{C}_x)$ by considering $s(x\partial_x)^l$, $0 \leq l \leq n-1$. Thus $\mathcal{M}(\mathbb{C}_x) = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}(\mathbb{C}_x)_{\alpha+n}$. x and ∂_x restrict to isomorphisms between $\mathcal{M}(\mathbb{C}_x)_{\alpha+n+1}$ and $\mathcal{M}(\mathbb{C}_x)_{\alpha+n}$. It follows $\mathcal{M}(\mathbb{C}_x) = 0$ or $\mathcal{M}(\mathbb{C}_x) = j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}(\mathbb{C}_x)$. Repeating the argument for \mathbb{C}_z we find $\mathcal{M} = 0$ or $\mathcal{M} = j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$. We have shown that $j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$ is simple. Further it is clear that $j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_x} \cong \mathcal{D}_{\mathbb{C}_x}/(x\partial_x - \alpha)\mathcal{D}_{\mathbb{C}_x}$. It follows with $j_! = \mathbb{D} j_* \mathbb{D}$, remark 2.1 and remark 2.3 (2) that $j_!\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_x} \cong \mathcal{D}_{\mathbb{C}_x}/(x\partial_x - \alpha)\mathcal{D}_{\mathbb{C}_x}$. Thus $j_!\Omega_{\mathbb{C}^\times}^{(\alpha)} \cong j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$. Recall that the canonical map $\phi : j_!\Omega_{\mathbb{C}^\times}^{(\alpha)} \rightarrow j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$ restricts to the identity on \mathbb{C}^\times . Thus $\ker \phi$ is supported on $(x=0) \sqcup (z=0)$. Since we know that $j_!\Omega_{\mathbb{C}^\times}^{(\alpha)}$ is simple it follows $\ker \phi = 0$. Because $j_!\Omega_{\mathbb{C}^\times}^{(\alpha)} \cong j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}$ it follows that ϕ also surjects.

(2). By definition we have $j_{!x,z}\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_z} \cong j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_z}$ and $j_{!x,z}\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_x} \cong j_!\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_x}$ restricting to the identity of $\Omega_{\mathbb{C}^\times}^{(\alpha)}$ on \mathbb{C}^\times . Further, by (1) we have an isomorphism $j_!\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_x} \xrightarrow{\cong} j_*\Omega_{\mathbb{C}^\times}^{(\alpha)}|_{\mathbb{C}_x}$ restricting to the identity of $\Omega_{\mathbb{C}^\times}^{(\alpha)}$ on \mathbb{C}^\times . The assertion follows. \square

As a consequence of this lemma we are left with determining the subquotients of the direct images of $\Omega_{\mathbb{C}^\times}$.

2.6.1. *Subquotients of $j_*\Omega_{\mathbb{C}^\times}$ and $j_!\Omega_{\mathbb{C}^\times}$.* $j_*\Omega_{\mathbb{C}^\times}$ fits into the exact sequence

$$0 \rightarrow j_{x*}\Omega_{\mathbb{C}_x} \rightarrow j_*\Omega_{\mathbb{C}^\times} \rightarrow \iota_{z*}\mathbb{C} \rightarrow 0 \quad (2.10)$$

and into the one obtained from it by interchanging x and z . Combining this with (2.5) we obtain a diagram of embeddings $A \hookrightarrow B$ such that B/A is simple

$$\begin{array}{ccc} & j_*\Omega_{\mathbb{C}^\times} & \\ \nearrow & & \nwarrow \\ j_{x*}\Omega_{\mathbb{C}_x} & & j_{z*}\Omega_{\mathbb{C}_z} \\ \nwarrow & & \nearrow \\ & \Omega_{\mathbb{P}^1} & \end{array}$$

Thus the simple objects $\Omega_{\mathbb{P}^1}$, $\iota_{x*}\mathbb{C}$ and $\iota_{z*}\mathbb{C}$ each have multiplicity one in $j_*\Omega_{\mathbb{C}^\times}$. This describes the subquotient structure of $j_*\Omega_{\mathbb{C}^\times}$. Applying \mathbb{D} to (2.10) yields

$$0 \leftarrow j_{x!}\Omega_{\mathbb{C}_x} \leftarrow j_!\Omega_{\mathbb{C}^\times} \leftarrow \iota_{z*}\mathbb{C} \leftarrow 0. \quad (2.11)$$

Combining this with (2.6) we see that the diagram of embeddings

$$\begin{array}{ccc} & j_!\Omega_{\mathbb{C}^\times} & \\ \uparrow & & \\ \iota_{x*}\mathbb{C} \oplus \iota_{z*}\mathbb{C} & & \\ \nearrow & & \nwarrow \\ \iota_{x*}\mathbb{C} & & \iota_{z*}\mathbb{C} \end{array}$$

has simple quotients.

2.6.2. *Subquotients of $j_{!x,z}\Omega_{\mathbb{C}^\times}$.* From the definition of $j_{!x,z}\Omega_{\mathbb{C}^\times}$ and § 2.6.1 we find the exact sequences

$$\begin{aligned} 0 \rightarrow j_{z!}\Omega_{\mathbb{C}_z} \rightarrow j_{!x,z}\Omega_{\mathbb{C}^\times} \rightarrow \iota_{x*}\mathbb{C} \rightarrow 0 \\ 0 \rightarrow \iota_{z*}\mathbb{C} \rightarrow j_{!x,z}\Omega_{\mathbb{C}^\times} \rightarrow j_{x*}\Omega_{\mathbb{C}_x} \rightarrow 0. \end{aligned} \quad (2.12)$$

We have a diagram of embeddings

$$\iota_{z*}\mathbb{C} \hookrightarrow j_{z!}\Omega_{\mathbb{C}_z} \hookrightarrow j_{!x,z}\Omega_{\mathbb{C}^\times}$$

with simple quotients. Again, $\Omega_{\mathbb{P}^1}$, $\iota_{x*}\mathbb{C}$ and $\iota_{z*}\mathbb{C}$ each have multiplicity one in $j_{!x,z}\Omega_{\mathbb{C}^\times}$.

2.7. \mathbb{G}_m -Equivariance. The multiplicative group \mathbb{G}_m acts on \mathbb{P}^1 by $t[z_0 : z_1] = [tz_0 : t^{-1}z_1]$. The orbits are $(x = 0)$, $(z = 0)$ and \mathbb{C}^\times . We describe the equivariance of the constructed \mathcal{D} -modules w.r.t. this action in the sense of [BB93][section 1.8]. We follow the terminology of [Gai05] and remark that in [Kas08] weak equivariance is called quasi-equivariance. To this end we note that if $Y = \mathbb{C}^\times, \mathbb{C}_x$ or \mathbb{P}^1 then \mathcal{D}_Y is naturally a weakly \mathbb{G}_m -equivariant \mathcal{D}_Y -module. If $\mathcal{I} \subseteq \mathcal{D}_Y$ is a sheaf of right ideals, then $\mathcal{D}_Y/\mathcal{I}$ has an induced

weakly equivariant structure if and only if $t\mathcal{I} \subseteq \mathcal{I}$ for all $t \in \mathbb{G}_m$. Also Ω_Y has a natural strongly \mathbb{G}_m -equivariant structure. Let \mathcal{M} be any holonomic \mathcal{D}_Y -module.

2.7.1. Equivariance of the holonomic dual. It is shown, in greater generality, in [RS13][Proposition 2.18] that there is an induced weakly \mathbb{G}_m -equivariant structure on $\mathbb{D}\mathcal{M}$ for any weakly \mathbb{G}_m -equivariant holonomic \mathcal{D}_Y -module \mathcal{M} . Further, it is stated in [BB93][section 2.5.8] that \mathbb{D} is a duality on the category of strongly \mathbb{G}_m -equivariant holonomic \mathcal{D} -modules (put $H = 1$ in loc. cit.).

2.7.2. Equivariance of $\Omega_{\mathbb{C}^\times}^{(\alpha)}$. Consider e.g. $Y = \mathbb{C}^\times$. From the above we conclude that $\Omega_{\mathbb{C}^\times}^{(\alpha)}$ is weakly \mathbb{G}_m -equivariant and for $\alpha = 0$ strongly \mathbb{G}_m -equivariant. The isomorphism

$$\Omega_{\mathbb{C}^\times} \xrightarrow{\cong} \Omega_{\mathbb{C}^\times}^{(0)} = \mathcal{D}_{\mathbb{C}^\times} / x\partial_x \mathcal{D}_{\mathbb{C}^\times}, \quad \frac{dx}{x} \mapsto \bar{1},$$

respects the equivariant structures. As the \mathbb{G}_m -equivariant irreducible local systems on \mathbb{C}^\times are in bijection with irreducible representations of the stabilizer $\{\pm 1\}$ of the \mathbb{G}_m -action at any point of \mathbb{C}^\times [BB81] we in fact also have a strongly \mathbb{G}_m -equivariant structure on $\Omega_{\mathbb{C}^\times}^{(1/2+n)}$, $n \in \mathbb{Z}$. (The latter is however not induced by the weakly equivariant structure on $\mathcal{D}_{\mathbb{C}^\times}$.)

2.7.3. Equivariance of direct images. The direct image $j_*\mathcal{M}$ has an induced weakly resp. strongly \mathbb{G}_m -equivariant structure if \mathcal{M} has a weakly resp. strongly \mathbb{G}_m -equivariant structure according to [Kas08][section 3.5] resp. [BB81]. By the same token $j_{x!}\Omega_{\mathbb{C}_x}$ and $\iota_{x*}\mathbb{C}$ have induced strongly \mathbb{G}_m -equivariant structures. By § 2.7.1 we obtain a weakly or strongly \mathbb{G}_m -equivariant structure on $j_!\mathcal{M}$ and a strongly \mathbb{G}_m -equivariant structure on $j_{x!}\Omega_{\mathbb{C}_x}$. Finally, we also have a weakly or strongly \mathbb{G}_m -equivariant structure on $j_{!x,z}\mathcal{M}$.

3. GENERALITIES ON \mathfrak{sl}_2 -MODULES

The standard basis of \mathfrak{sl}_2 will be denoted by e, f, h . For $\lambda \in \mathbb{C}$ we have the usual \mathfrak{sl}_2 -module $M(\lambda)$, the *Verma module of highest weight λ* , and $L(\lambda)$, the *simple \mathfrak{sl}_2 -module of highest weight λ* . In accordance with our convention for \mathcal{D} -modules we will however work with the corresponding right \mathfrak{sl}_2 -modules obtained from them via the anti-involution of \mathfrak{sl}_2 given by $e \mapsto e, f \mapsto f, h \mapsto -h$. Thus $M(\lambda)$ will have h -weights bounded from below, namely $-\lambda + 2\mathbb{Z}_{\geq 0}$. Below, the notation $L(\lambda)$ will only be used when $L(\lambda)$ is finite dimensional, i.e. in the case $\lambda \in \mathbb{Z}_{\geq 0}$. More generally, we consider *weight modules* for \mathfrak{sl}_2 . These are \mathfrak{sl}_2 -modules M such that $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$, where $M_\lambda = \{v \in M \mid vh = \lambda v\}$ is the *(h-)weight space for the weight λ* . Let $T \subseteq \mathrm{SL}_2$ be the diagonal matrices.

Remark 3.1. Let M be a \mathfrak{sl}_2 -module. Then the $\mathrm{Lie} T = \mathbb{C}h$ -action on M integrates to a (algebraic) T -action on M if and only if M is a weight module with weights in \mathbb{Z} , see e.g. [Jan][I, section 2.11].

3.1. Duality $(\cdot)^\vee$. If $\dim M_\lambda < \infty$ for all λ we define the *dual of M* as $M^\vee = \bigoplus_{\lambda \in \mathbb{C}} \text{Hom}_{\mathbb{C}}(M_\lambda, \mathbb{C})$ by letting $X \in \mathfrak{sl}_2$ act on $\phi \in M^\vee$ by $(\phi X)(v) = \phi(v\tau(X))$. Here τ is the anti-involution of \mathfrak{sl}_2 defined by $h \mapsto h$, $e \mapsto f$, $f \mapsto e$. Then M^\vee is again a weight module and $(M^\vee)_\lambda = \text{Hom}_{\mathbb{C}}(M_\lambda, \mathbb{C})$. Thus, if M is e.g. a highest weight module, then so is M^\vee . We record for later use that $M(\lambda) \cong M(\lambda)^\vee$ if and only if $\lambda \notin \mathbb{Z}_{\geq 0}$.

4. H^0 AND H^1

Let us abbreviate the sheaf cohomology groups $H^k(\mathbb{P}^1, \cdot)$ by $H^k(\cdot)$. Let us from now on drop the subscripts \mathbb{P}^1 in $\mathcal{D}_{\mathbb{P}^1}$, $\mathcal{O}_{\mathbb{P}^1}$, $\Omega_{\mathbb{P}^1}$, etc. to lighten notation. Consider the SL_2 -action on \mathbb{P}^1 given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} [z_0 : z_1] = [az_0 + bz_1 : cz_0 + dz_1]$. (The action of the subgroup $T \cong \mathbb{G}_m$ coincides with the one considered in § 2.7.) It induces an isomorphism of Lie algebras $\mathfrak{sl}_2 \xrightarrow{\cong} H^0(\mathcal{T})$, where \mathcal{T} is the tangent sheaf of \mathbb{P}^1 , that maps the standard basis

$$e \mapsto \partial_x = -z^2 \partial_z, \quad h \mapsto -2x \partial_x = 2z \partial_z, \quad f \mapsto -x^2 \partial_x = \partial_z. \quad (4.1)$$

(= here abbreviates equality after restriction to \mathbb{C}^\times and the global section of \mathcal{T} is of course obtained by gluing.) It is known [BB81] that this induces an isomorphism $(\mathcal{U}\mathfrak{sl}_2)_0 := \mathcal{U}\mathfrak{sl}_2/(c) \xrightarrow{\cong} H^0(\mathcal{D})$ of algebras. Here $\mathcal{U}\mathfrak{sl}_2$ is the universal enveloping algebra of \mathfrak{sl}_2 and c is the Casimir central element $c = ef + fe + \frac{h^2}{2}$. Further, we denote here and below by (\cdot) the right ideal generated by \cdot in an associative algebra. Thus, if \mathcal{M} is a (right) \mathcal{D} -module, then $H^0(\mathcal{M})$ is a right module over $(\mathcal{U}\mathfrak{sl}_2)_0$. Recall that we may compute $H^k(\mathcal{F})$ for any quasicohherent \mathcal{O} -module \mathcal{F} by the Čech complex of an open affine cover [Har77][III, Theorem 4.5]. For the cover $\{\mathbb{C}_x, \mathbb{C}_z\}$ the Čech complex is

$$\mathcal{F}(\mathbb{C}_x) \oplus \mathcal{F}(\mathbb{C}_z) \rightarrow \mathcal{F}(\mathbb{C}^\times), \quad (s_1, s_2) \mapsto s_1|_{\mathbb{C}^\times} - s_2|_{\mathbb{C}^\times}, \quad (4.2)$$

under the usual identification $\mathbb{C}_x \cap \mathbb{C}_z = \mathbb{C}^\times$. If \mathcal{F} in addition has the structure of a \mathcal{D} -module, then (4.2) is H^0 of a morphism of \mathcal{D} -modules

$$\mathcal{C}^0(\mathcal{F}) := j_x \cdot j_x^{-1} \mathcal{F} \oplus j_z \cdot j_z^{-1} \mathcal{F} \rightarrow \mathcal{C}^1(\mathcal{F}) := j \cdot j^{-1} \mathcal{F}. \quad (4.3)$$

Hence $H^1(\mathcal{F})$ is a $H^0(\mathcal{D})$ -module. E.g. for $\mathcal{F} = \Omega$ (4.2) is

$$\mathbb{C}[x] dx \oplus \mathbb{C}[z] dz \rightarrow \mathbb{C}[x, x^{-1}] dx, \quad (x^n dx, z^m dz) \mapsto (x^n + x^{-m-2}) dx.$$

Thus

$$H^0(\Omega) = 0 \quad H^1(\Omega) = \mathbb{C} x^{-1} dx \cong L(0). \quad (4.4)$$

This calculation can also be found in [Har77][III, Example 4.0.3]. Similarly we find

$$H^0(j_x \cdot \Omega_{\mathbb{C}_x}) = \Omega_{\mathbb{C}_x}(\mathbb{C}_x) \cong M(-2) \quad H^1(j_x \cdot \Omega_{\mathbb{C}_x}) = 0. \quad (4.5)$$

Further, we find with $\iota_{x*} \mathbb{C}|_{\mathbb{C}_z} = \mathcal{D}_{\mathbb{C}_z}/z\mathcal{D}_{\mathbb{C}_z}$

$$H^0(\iota_{x*} \mathbb{C}) = \mathbb{C}[\overline{\partial_z}] \cong M(0) \quad H^1(\iota_{x*} \mathbb{C}) = 0. \quad (4.6)$$

Further, we find with $j_{x!}\Omega_{\mathbb{C}_x}|\mathbb{C}_z = \mathcal{D}_{\mathbb{C}_z}/\partial_z z \mathcal{D}_{\mathbb{C}_z}$, see (2.7),

$$H^0(j_{x!}\Omega_{\mathbb{C}_x}) = \overline{\partial_z} \mathbb{C}[\overline{\partial_z}] \cong M(0) \quad H^1(j_{x!}\Omega_{\mathbb{C}_x}) = \mathbb{C} x^{-1} dx \cong L(0). \quad (4.7)$$

We will denote by $(\mathcal{U}\mathfrak{sl}_2)_0 \text{ mod}$ and $\text{mod}(\mathcal{U}\mathfrak{sl}_2)_0$ the category of (not necessarily finitely generated) left resp. right $(\mathcal{U}\mathfrak{sl}_2)_0$ -modules.

4.1. Auto-Equivalences $(\cdot)^-$. Before proceeding in the computation of the cohomology it is convenient to write out the operations induced by exchanging the coordinate x with z . Consider the involution I of \mathbb{P}^1 given by $x \mapsto x^{-1}$. If \mathcal{M} is \mathcal{D} -module, we define \mathcal{M}^- as the direct image by I , $\mathcal{M}^- = I_*\mathcal{M} = I_*(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}_I)$. Here $\mathcal{D}_I = \mathcal{O} \otimes_{I^{-1}\mathcal{O}} I^{-1}\mathcal{D}$ is the transfer \mathcal{D} - $I^{-1}\mathcal{D}$ -bimodule, see e.g. [Kas00][section 4.1]. Since I is a proper morphism, $(\cdot)^-$ commutes with \mathbb{D} , see [Ber][p. 19].

Remark 4.1. Let \mathcal{R} be a sheaf of \mathbb{C} -algebras on a topological space X . Let \mathcal{M} be a right \mathcal{R} -module. Let \mathcal{F} be a free (not just locally free) left \mathcal{R} -module of rank one. For $U \subseteq X$ open we have isomorphisms of vector spaces

$$\mathcal{M}(U) \otimes_{\mathcal{R}(U)} \mathcal{F}(U) \cong \mathcal{M}(U)$$

compatible with the restriction maps. Thus, the sheafification maps $\mathcal{M}(U) \otimes_{\mathcal{R}(U)} \mathcal{F}(U) \rightarrow (\mathcal{M} \otimes_{\mathcal{R}} \mathcal{F})(U)$, which we have by definition of $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{F}$, are isomorphisms.

Lemma 4.1. Let $U \subseteq \mathbb{P}^1$ be open. The left action of \mathcal{D} on \mathcal{D}_I induces an isomorphism

$$l : \mathcal{D}(U) \xrightarrow{\cong} \mathcal{O}(U) \otimes_{(I^{-1}\mathcal{O})(U)} (I^{-1}\mathcal{D})(U) \xrightarrow{\cong} \mathcal{D}_I(U),$$

where the first map is $p \mapsto p(1 \otimes 1)$. The right action of $I^{-1}\mathcal{D}$ on \mathcal{D}_I induces an isomorphism

$$r : \mathcal{D}(I(U)) \xrightarrow{\cong} \mathcal{O}(U) \otimes_{(I^{-1}\mathcal{O})(U)} (I^{-1}\mathcal{D})(U) \xrightarrow{\cong} \mathcal{D}_I(U),$$

where the first map is $p \mapsto 1 \otimes p$. The compositions $r^{-1}l$ define an isomorphism of sheaves of algebras $J : \mathcal{D} \xrightarrow{\cong} I^{-1}\mathcal{D}$ such that $(I^{-1}J)J = \text{id}_{\mathcal{D}}$.

Proof. Remark 4.1 shows that the second map in the definition of l and r is an isomorphism since $I^\sharp : I^{-1}\mathcal{O} \rightarrow \mathcal{O}$ is an isomorphism of sheaves of algebras. For the remaining statements one recalls [Kas00][section 4.1] the definition of the left action of \mathcal{D} on \mathcal{D}_I . The differential of I induces an isomorphism of sheaves of Lie algebras $\mathcal{T} \xrightarrow{\cong} I^{-1}\mathcal{T}$. This map together with $(I^\sharp)^{-1} : \mathcal{O} \xrightarrow{\cong} I^{-1}\mathcal{O}$ induces an isomorphism of sheaves of algebras $\mathcal{D} \xrightarrow{\cong} I^{-1}\mathcal{D}$, which is J . \square

Remark 4.2. From (4.1) we see that under the isomorphism $\mathfrak{sl}_2 \xrightarrow{\cong} H^0(\mathcal{T})$ the map $\mathcal{T} \rightarrow I^{-1}\mathcal{T}$ of the proof of Lemma 4.1 is given by $h \mapsto -h$, $e \mapsto f$, $f \mapsto e$.

J induces an involution of $(\mathcal{U}\mathfrak{sl}_2)_0$. We will denote the twist of a $(\mathcal{U}\mathfrak{sl}_2)_0$ -module M by this involution by M^- . If M is a lowest weight module, then by remark 4.2 M^- is a highest weight module and vice versa. $(\cdot)^-$ defines an exact auto-equivalence of the

category $\text{mod } \mathcal{D}$ and $\text{mod}(\mathcal{U}\mathfrak{sl}_2)_0$ respectively. We have $\tau J = J\tau$ and hence $(\cdot)^-$ commutes with $(\cdot)^\vee$. Here τ and $(\cdot)^\vee$ were defined in § 3.1.

Lemma 4.2. There are natural isomorphisms $H^k(\mathcal{M}^-) \cong H^k(\mathcal{M})^-$ for any $\mathcal{M} \in \text{mod } \mathcal{D}$.

Proof. We have $H^k(I_*\mathcal{M}) \cong H^k(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}_I)$ since I is an affine morphism. We reason with the complex (4.3). We have an isomorphism $H^0(\mathcal{C}^\bullet(\mathcal{M})) \otimes_{H^0(\mathcal{D})} H^0(\mathcal{D}_I) \xrightarrow{\cong} H^0(\mathcal{C}^\bullet(\mathcal{M}) \otimes_{\mathcal{D}} \mathcal{D}_I)$ of complexes due to remark 4.1. Further, we claim that we have an isomorphism of complexes $H^0(\mathcal{C}^\bullet(\mathcal{M}) \otimes_{\mathcal{D}} \mathcal{D}_I) \cong H^0(\mathcal{C}^\bullet(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}_I))$. Indeed, this follows from the fact that for any open embedding κ into \mathbb{P}^1 we have $(\kappa.\kappa^{-1}\mathcal{M}) \otimes_{\mathcal{D}} \mathcal{D}_I \cong \kappa.\kappa^{-1}(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}_I)$ as \mathcal{D} -modules, where \mathcal{D} acts via $\mathcal{D} \rightarrow \kappa.\kappa^{-1}\mathcal{D}$, as can be checked on open subsets. In this way we obtain an isomorphism of complexes $H^0(\mathcal{C}^\bullet(\mathcal{M})) \otimes_{H^0(\mathcal{D})} H^0(\mathcal{D}_I) \xrightarrow{\cong} H^0(\mathcal{C}^\bullet(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}_I))$. Lemma 4.1 together with the definition of $(\cdot)^-$ now implies the statement. \square

As an application of this lemma we e.g. deduce from $(j_x.\Omega_{\mathbb{C}_x})^- \cong j_z.\Omega_{\mathbb{C}_z}$ and (4.5) that

$$H^0(j_z.\Omega_{\mathbb{C}_z}) = \Omega_{\mathbb{C}_z}(\mathbb{C}_z) \cong M(-2)^- \quad H^1(j_z.\Omega_{\mathbb{C}_z}) = 0 ,$$

but of course this can also be computed directly.

4.2. H^0 and H^1 of $j.\Omega_{\mathbb{C}^\times}^{(\alpha)}$, $\alpha \notin \mathbb{Z}$, and definition of $R(\alpha)$. According to (2.9) the restriction $(j.\Omega_{\mathbb{C}^\times}^{(\alpha)})(\mathbb{C}_x) \rightarrow (j.\Omega_{\mathbb{C}^\times}^{(\alpha)})(\mathbb{C}^\times)$ is an isomorphism of vector spaces. Thus $H^1(j.\Omega_{\mathbb{C}^\times}^{(\alpha)}) = 0$. Further, according to (2.9) and the corresponding statement for \mathbb{C}_z

$$H^0(j.\Omega_{\mathbb{C}^\times}^{(\alpha)}) = H^0(\mathbb{C}^\times, \Omega_{\mathbb{C}^\times}^{(\alpha)}) \cong (\mathcal{U}\mathfrak{sl}_2)_0 / (h + 2\alpha) .$$

We set

$$R(\alpha) = (\mathcal{U}\mathfrak{sl}_2)_0 / (h + 2\alpha) = \mathcal{U}\mathfrak{sl}_2 / (h + 2\alpha, ef + \alpha(\alpha + 1)) . \quad (4.8)$$

The PBW theorem for $\mathcal{U}\mathfrak{sl}_2$ implies $R(\alpha) = \bar{f}\mathbb{C}[\bar{f}] \oplus \mathbb{C}[\bar{e}]$ as a vector space. Due to remark 2.3 (2) and the fact that the h -weights of $R(\alpha)$ are $-2\alpha + 2\mathbb{Z}$ we have $R(\alpha) \cong R(\beta)$ if and only if $\alpha - \beta \in \mathbb{Z}$. The following lemma and its proof is analogous to Lemma 2.1.

Lemma 4.3. $R(\alpha)$ is a simple \mathfrak{sl}_2 -module for $\alpha \notin \mathbb{Z}$.

Proof. Let $M \neq 0$ be a submodule of $R(\alpha)$. As in the proof of Lemma 2.1 we show that M is a direct sum of its h -weight spaces $M = \bigoplus_{n \in \mathbb{Z}} M_{-2\alpha+2n}$ with $M_{-2\alpha+2n} \neq 0$ for some n . We have linear maps

$$M_{-2\alpha+2(n+1)} \xrightleftharpoons[\cdot f]{\cdot e} M_{-2\alpha+2n} .$$

From the definition (4.8) it follows

$$\cdot ef = (n - \alpha)(\alpha - n - 1) \text{id}_{M_{-2\alpha+2(n+1)}} \quad \cdot fe = (n - \alpha)(\alpha - n - 1) \text{id}_{M_{-2\alpha+2n}} . \quad (4.9)$$

Thus, since $\alpha \notin \mathbb{Z}$, the maps e and f are invertible for each $n \in \mathbb{Z}$. It follows $M_{-2\alpha+2n} \neq 0$ for all $n \in \mathbb{Z}$ and hence $M = R(\alpha)$. \square

4.3. H^0 and H^1 of $j_*\Omega_{\mathbb{C}^\times}$. We find $H^0(j_*\Omega_{\mathbb{C}^\times}) = \Omega_{\mathbb{C}^\times}(\mathbb{C}^\times) \cong R(0)$, now allowing $\alpha = 0$ in the definition (4.8). Also $H^1(j_*\Omega_{\mathbb{C}^\times}) = 0$.

Remark 4.3. For any $\alpha \in \mathbb{C}$ we have $(j_*\Omega_{\mathbb{C}^\times}^{(\alpha)})^- \cong j_*\Omega_{\mathbb{C}^\times}^{(-\alpha)}$ and by Remark 4.2 $R(\alpha)^- \cong R(-\alpha)$.

Since $H^1(j_{x*}\Omega_{\mathbb{C}_x}) = 0$ the long exact sequence of cohomology of (2.10) gives the exact sequence

$$0 \rightarrow M(-2) \rightarrow R(0) \rightarrow M(0)^- \rightarrow 0.$$

Applying $(\cdot)^-$ we obtain the exact sequence

$$0 \rightarrow M(-2)^- \rightarrow R(0) \rightarrow M(0) \rightarrow 0.$$

4.4. H^0 and H^1 of $j_!\Omega_{\mathbb{C}^\times}$. From $j_!\Omega_{\mathbb{C}^\times}|_{\mathbb{C}_z} \cong j_{x!}\Omega_{\mathbb{C}_x}|_{\mathbb{C}_z} \cong \mathcal{D}_{\mathbb{C}_z}/\partial_z z \mathcal{D}_{\mathbb{C}_z}$, see (2.7), and the corresponding statement for \mathbb{C}_x we deduce $H^0(j_!\Omega_{\mathbb{C}^\times}) \cong M(0) \oplus M(0)^-$. Also $H^1(j_!\Omega_{\mathbb{C}^\times}) = x^{-1}dx \cong L(0)$. Since $H^1(\iota_{z*}\mathbb{C}) = 0$ the long exact sequence of cohomology of (2.11) gives the exact sequence

$$0 \leftarrow M(0) \leftarrow M(0) \oplus M(0)^- \leftarrow M(0)^- \leftarrow 0.$$

4.5. H^0 and H^1 of $j_{!x,z}\Omega_{\mathbb{C}^\times}$ and $j_{x!z}\Omega_{\mathbb{C}^\times}$. As in § 4.3 and § 4.4 we find $H^0(j_{!x,z}\Omega_{\mathbb{C}^\times}) \cong R(-1)$ and $H^1(j_{!x,z}\Omega_{\mathbb{C}^\times}) = 0$ as well as $H^0(j_{x!z}\Omega_{\mathbb{C}^\times}) \cong R(1)$ and $H^1(j_{x!z}\Omega_{\mathbb{C}^\times}) = 0$. Again, one can write down the long exact sequence of cohomology of (2.12).

Remark 4.4. We compare the above results for H^0 and H^1 with the equivalence of Beilinson-Bernstein for *left* \mathcal{D} -modules [BB81]: Then $H^1 = 0$ and H^0 is an exact equivalence of categories $\mathcal{D} \text{ mod} \rightarrow (\mathcal{U}\mathfrak{sl}_2)_0 \text{ mod}$. The functor $\Delta: (\mathcal{U}\mathfrak{sl}_2)_0 \text{ mod} \rightarrow \mathcal{D} \text{ mod}$ [BB81] defined by $\Delta(M) = \mathcal{D} \otimes_{(\mathcal{U}\mathfrak{sl}_2)_0} \underline{M}$, where \underline{V} denotes the constant sheaf on \mathbb{P}^1 constructed from the vector space V , is quasi-inverse to H^0 . Namely, the natural morphism of \mathcal{D} -modules

$$\mathcal{D} \otimes_{(\mathcal{U}\mathfrak{sl}_2)_0} \underline{H^0(\mathcal{M})} \rightarrow \mathcal{M} \tag{4.10}$$

is an isomorphism of \mathcal{D} -modules. Note that a \mathcal{D} -module is generated by its global sections if and only if (4.10) surjects. Of the above \mathcal{D} -modules, only $j_{x*}\Omega_{\mathbb{C}_x}$ and $\iota_{x*}\mathbb{C}$ satisfy this. In fact, $j_{x*}\Omega_{\mathbb{C}_x}$ is generated by its global sections even as an \mathcal{O} -module. In the above H^0 and H^1 the dual Verma module $M(0)^\vee$ does not appear and $M(0)$ is H^0 both of $\iota_{x*}\mathbb{C}$ and $j_{x!}\Omega_{\mathbb{C}_x}$. In fact, by [BB93][3.3.3 Corollary] the equivalence $H^0: \mathcal{D} \text{ mod} \rightarrow (\mathcal{U}\mathfrak{sl}_2)_0 \text{ mod}$ restricts to an equivalence between the subcategories of strongly \mathbb{G}_m -equivariant \mathcal{D} -modules and of $(\mathcal{U}\mathfrak{sl}_2)_0$ -modules on which the $\mathbb{C}h$ -action comes from a T -action. Coming back to our computations of H^0 , it is manifest, cf. Remark 3.1, that the direct images which are strongly \mathbb{G}_m -equivariant as discussed in § 2.7.3 have the property that their H^0 carry a $\mathbb{C}h$ -action coming from a T -action.

5. TWISTED \mathcal{D} -MODULES

5.1. Generalities on twisted \mathcal{D} -modules. Let $\lambda \in \mathbb{Z}$. We have a corresponding line bundle $\mathcal{O}(\lambda)$ on \mathbb{P}^1 . The sheaf of algebras $\mathcal{D}(\lambda) := \mathcal{O}(-\lambda) \otimes_{\mathcal{O}} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{O}(\lambda)$ is the sheaf of $\mathcal{O}(\lambda)$ -twisted differential operators [BB81], [BB93][section 2.1]. E.g. $\Omega \otimes_{\mathcal{O}} \mathcal{O}(\lambda)$ is naturally a (right) $\mathcal{D}(\lambda)$ -module. $\mathcal{D}(\lambda)$ possesses an increasing filtration $F^n \mathcal{D}(\lambda)$, $n \in \mathbb{Z}_{\geq 0}$, by locally free \mathcal{O} -submodules of finite rank defined by the order of the differential operator. Let us first recall a description of $F^1 \mathcal{D}(\lambda)$ given in [BB93][section 2.1].

Remark 5.1. The commutator $[\cdot, \cdot]$ in $\mathcal{D}(\lambda)$ turns $F^1 \mathcal{D}(\lambda)$ into a sheaf of Lie algebras. There is a short exact sequence of locally free \mathcal{O} -modules

$$0 \rightarrow F^0 \mathcal{D}(\lambda) = \mathcal{O} \rightarrow F^1 \mathcal{D}(\lambda) \rightarrow \mathcal{T} \rightarrow 0, \quad (5.1)$$

where $F^1 \mathcal{D}(\lambda) \rightarrow \mathcal{T} = \mathcal{D}er(\mathcal{O})$, $p \mapsto [p, \cdot]$. Here $\mathcal{D}er(\mathcal{O})$ is the \mathcal{O} -module of derivations. This makes $F^1 \mathcal{D}(\lambda)$ into a Picard Lie algebroid in the sense of [BB93][section 2.1], a special kind of Lie algebroid.

Remark 5.2. $F^1 \mathcal{D}(-\lambda)$ identifies with the Atiyah algebroid of the principal \mathbb{G}_m -bundle $\mathcal{O}(\lambda)^\times$ associated to $\mathcal{O}(\lambda)$. Let us elaborate on this. The *Atiyah algebroid* of $\mathcal{O}(\lambda)^\times$ is given by an exact sequence

$$0 \rightarrow \text{Lie } \mathbb{G}_m \times_{\mathbb{G}_m} \mathcal{O}(\lambda)^\times \rightarrow (p.\mathcal{T}_{\mathcal{O}(\lambda)^\times})^{\mathbb{G}_m} \rightarrow \mathcal{T} \rightarrow 0, \quad (5.2)$$

where $p : \mathcal{O}(\lambda)^\times \rightarrow \mathbb{P}^1$ denotes the projection and $(\cdot)^{\mathbb{G}_m}$ \mathbb{G}_m -invariants. The first map in (5.2) realizes the adjoint bundle $\text{Lie } \mathbb{G}_m \times_{\mathbb{G}_m} \mathcal{O}(\lambda)^\times \cong \mathcal{O}$ as the vertical vector fields on the total space of the \mathbb{G}_m -bundle, which is also denoted by $\mathcal{O}(\lambda)^\times$. We have an isomorphism $p.\mathcal{O}_{\mathcal{O}(\lambda)^\times} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(\lambda n)$ of \mathbb{Z} -graded sheaves of $\mathcal{O}_{\mathbb{P}^1}$ -algebras. Here the homogeneous component $\mathcal{O}(\lambda n)$ identifies with the subsheaf on which the vertical vector fields act by multiplication by n . $(p.\mathcal{T}_{\mathcal{O}(\lambda)^\times})^{\mathbb{G}_m}$ acts on $p.\mathcal{O}_{\mathcal{O}(\lambda)^\times}$ preserving the homogeneous components. In this way we obtain a map $(p.\mathcal{T}_{\mathcal{O}(\lambda)^\times})^{\mathbb{G}_m} \rightarrow F^1 \mathcal{D}(-\lambda)$, which turns out to be an isomorphism of Lie algebroids.

We have a map of Lie algebroids

$$\text{ac} : \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \rightarrow F^1 \mathcal{D}(\lambda) \quad (5.3)$$

induced by the natural SL_2 -equivariant structure of $\mathcal{O}(\lambda)$. This makes precise the statement that ac lifts the infinitesimal action of SL_2 on \mathbb{P}^1 to first order differential operators. The expression z_0^λ respectively z_1^λ is a nowhere vanishing section of $\mathcal{O}(\lambda)$ on \mathbb{C}_x respectively \mathbb{C}_z . On global sections we have

$$\begin{aligned} \text{ac}(e \otimes 1) &= z_0^{-\lambda} \otimes \partial_x \otimes z_0^\lambda = z_1^{-\lambda} \otimes (-z^2 \partial_z - \lambda z) \otimes z_1^\lambda \\ \text{ac}(h \otimes 1) &= z_0^{-\lambda} \otimes (-2x \partial_x - \lambda) \otimes z_0^\lambda = z_1^{-\lambda} \otimes (2z \partial_z + \lambda) \otimes z_1^\lambda \\ \text{ac}(f \otimes 1) &= z_0^{-\lambda} \otimes (-x^2 \partial_x - \lambda x) \otimes z_0^\lambda = z_1^{-\lambda} \otimes \partial_z \otimes z_1^\lambda, \end{aligned} \quad (5.4)$$

cf. (4.1).

Remark 5.3. The space of extensions of the form (5.1) is $\text{Ext}^1(\mathcal{T}, \mathcal{O}) = H^1(\Omega^1) = \mathbb{C} \frac{dx}{x}$, see (4.4). It has the following Čech description. Let $\sigma_{x,z} : \mathcal{T}(\mathbb{C}_{x,z}) \rightarrow F^1\mathcal{D}(\lambda)(\mathbb{C}_{x,z})$ be $\mathcal{O}(\mathbb{C}_{x,z})$ -linear sections of (5.1) over $\mathbb{C}_{x,z}$. If σ'_x is another such section, then $\sigma_x - \sigma'_x : \mathcal{T}(\mathbb{C}_x) \rightarrow \mathcal{O}(\mathbb{C}_x)$ is an arbitrary $\mathcal{O}(\mathbb{C}_x)$ -linear map, i.e. an element of $\Omega^1(\mathbb{C}_x)$. The difference $\sigma_x|_{\mathbb{C}^\times} - \sigma_z|_{\mathbb{C}^\times} : \mathcal{T}(\mathbb{C}^\times) \rightarrow \mathcal{O}(\mathbb{C}^\times)$ is $\mathcal{O}(\mathbb{C}^\times)$ -linear and hence defines an element of $\Omega^1(\mathbb{C}^\times)$. The extension class is given by the class of $\sigma_x|_{\mathbb{C}^\times} - \sigma_z|_{\mathbb{C}^\times}$ in $H^1(\Omega^1)$, which is independent of the choice of $\sigma_{x,z}$. Let us compute the extension class from the formulae (5.4). The assignment $\partial_x \mapsto z_0^{-\lambda} \otimes \partial_x \otimes z_0^\lambda$ defines a unique σ_x and $\partial_z \mapsto z_1^{-\lambda} \otimes \partial_z \otimes z_1^\lambda$ defines a unique σ_z . By $\mathcal{O}(\mathbb{C}_z)$ -linearity σ_z sends $-z^2\partial_z \mapsto z_1^{-\lambda} \otimes (-z^2\partial_z) \otimes z_1^\lambda$ and consequently by the first line in (5.4) $\sigma_x|_{\mathbb{C}^\times} - \sigma_z|_{\mathbb{C}^\times}$ sends

$$\partial_x = -z^2\partial_z \mapsto z_1^{-\lambda} \otimes (-\lambda z) \otimes z_1^\lambda = -\lambda z.$$

Thus, the extension class is $-\lambda \frac{dx}{x}$, which is the first Chern class of $\mathcal{O}(-\lambda)$. This is a general fact, cf. [BB93][section 2.1].

(5.3) induces a map of sheaves of algebras $\mathcal{U}(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}) \rightarrow \mathcal{D}(\lambda)$, where $\mathcal{U}(\cdot)$ is defined in [BB93][section 1.2.5]. The corresponding map on global sections $\mathcal{U}\mathfrak{sl}_2 \rightarrow H^0(\mathcal{D}(\lambda))$ induces an isomorphism $(\mathcal{U}\mathfrak{sl}_2)_{\chi_\lambda} := \mathcal{U}\mathfrak{sl}_2 / (c - \frac{1}{2}\lambda(\lambda - 2)) \xrightarrow{\cong} H^0(\mathcal{D}(\lambda))$ [BB81]. Thus, if \mathcal{M} is a $\mathcal{D}(\lambda)$ -module, then the $H^j(\mathcal{M})$ are naturally $(\mathcal{U}\mathfrak{sl}_2)_{\chi_\lambda}$ -modules.

5.2. H^0 and H^1 . The results of this section generalize the ones of § 4. From the Čech complex (4.2) and (5.4) we find

Theorem 5.1.

$$\begin{aligned} H^0(\Omega \otimes_{\mathcal{O}} \mathcal{O}(\lambda)) &\cong \begin{cases} L(\lambda - 2) & \lambda \geq 2 \\ 0 & \lambda \leq 1 \end{cases} & H^1(\Omega \otimes_{\mathcal{O}} \mathcal{O}(\lambda)) &\cong \begin{cases} 0 & \lambda \geq 1 \\ L(-\lambda) & \lambda \leq 0 \end{cases} \\ H^0(j_{x*}\Omega_{\mathbb{C}_x}) &\cong M(\lambda - 2)^\vee & H^1(j_{x*}\Omega_{\mathbb{C}_x}) &= 0 \\ H^0(\iota_{x*}\mathbb{C}) &\cong M(-\lambda) & H^1(\iota_{x*}\mathbb{C}) &= 0 \\ H^0(j_{x!}\Omega_{\mathbb{C}_x}) &\cong \begin{cases} M(\lambda - 2) & \lambda \geq 1 \\ M(-\lambda) & \lambda \leq 0 \end{cases} & H^1(j_{x!}\Omega_{\mathbb{C}_x}) &\cong \begin{cases} 0 & \lambda \geq 1 \\ L(-\lambda) & \lambda \leq 0 \end{cases} \end{aligned} \quad . \quad \square \quad (5.5)$$

Even though this is not reflected in the notation, the direct images $j_{x*}\Omega_{\mathbb{C}_x}$ etc. are now of course understood as $\mathcal{D}(\lambda)$ -modules, differing from the previously considered modules by a factor $\mathcal{O}(\lambda)$. Theorem 5.1 is certainly well-known. In fact a similar statement is known to hold for any semisimple Lie algebra over \mathbb{C} instead of \mathfrak{sl}_2 . The first part is the Borel-Weil-Bott theorem, while the identification of the cohomology of $j_{x*}\Omega_{\mathbb{C}_x}$ and $\iota_{x*}\mathbb{C}$ is proven in [BK81][equation (5.1.2) and Corollary 5.8] and [Gai05][Theorem 10.6] in the untwisted case.

Remark 5.4. Independent of the twist it holds that H^0 of ι_{x*} resp. j_{x*} is a Verma resp. dual Verma module.

Let $\text{mod } \mathcal{D}(\lambda)$ denote the category of $\mathcal{D}(\lambda)$ -modules quasicoherent over \mathcal{O} . The definitions and results of § 4.1 carry over to the twisted case: Since $I^*\mathcal{O}(\lambda) \cong \mathcal{O}(\lambda)$ we again have an exact auto-equivalence $(\cdot)^-$ of $\text{mod } \mathcal{D}(\lambda)$, further an exact auto-equivalence $(\cdot)^-$ of $\text{mod}(\mathcal{U}\mathfrak{sl}_2)_{\chi_\lambda}$ and Lemma 4.2.

5.2.1. H^0 and H^1 of $j.\Omega_{\mathbb{C}^\times}^{(\alpha)}$, $\alpha \notin \mathbb{Z}$, and definition of $R(\lambda, \alpha)$. We find

Theorem 5.2.

$$H^0(j.\Omega_{\mathbb{C}^\times}^{(\alpha)}) \cong (\mathcal{U}\mathfrak{sl}_2)_{\chi_\lambda}/(h + 2\alpha + \lambda) \quad H^1(j.\Omega_{\mathbb{C}^\times}^{(\alpha)}) = 0.$$

□

We set

$$R(\lambda, \alpha) = (\mathcal{U}\mathfrak{sl}_2)_{\chi_\lambda}/(h + 2\alpha + \lambda) = \mathcal{U}\mathfrak{sl}_2/(h + 2\alpha + \lambda, ef + (\alpha + \lambda)(\alpha + 1)) \quad (5.6)$$

generalizing (4.8). The following lemma and its proof generalize Lemma 4.3 and hence we will be brief.

Lemma 5.1. Let $\alpha \notin \mathbb{Z}$. $R(\lambda, \alpha)$ is simple. $R(\lambda, \alpha) \cong R(\mu, \beta)$ if and only if $\mu \in \{\lambda, -\lambda + 2\}$ and $\alpha - \beta \in \mathbb{Z}$.

Proof. We conclude from (5.6) for any submodule M of $R(\lambda, \alpha)$

$$\begin{aligned} \cdot ef &= (n - \alpha)(\alpha - n - 1 + \lambda) \text{id}_{M_{-2\alpha - \lambda + 2(n+1)}} \\ \cdot fe &= (n - \alpha)(\alpha - n - 1 + \lambda) \text{id}_{M_{-2\alpha - \lambda + 2n}} \end{aligned} \quad (5.7)$$

generalizing (4.9). $R(\lambda, \alpha) \cong R(\mu, \beta)$ implies that the modules have the same central character. Thus $\mu \in \{\lambda, -\lambda + 2\}$. As the weights of $R(\lambda, \alpha)$ are $-2\alpha - \lambda + 2\mathbb{Z}$ we conclude $\alpha - \beta \in \mathbb{Z}$. □

5.2.2. *Description of $R(\lambda, \alpha)$, $\alpha \in \mathbb{Z}$.* Before proceeding to the computation of the remaining H^0 and H^1 , it is convenient to analyze $R(\lambda, \alpha)$ for $\alpha \in \mathbb{Z}$. We will find that for many values of the parameter α the $R(\lambda, \alpha)$ are isomorphic. Hence, we will introduce some notation for the isomorphism classes.

Remark 5.5. For any $\alpha \in \mathbb{C}$ $R(\lambda, \alpha)^- \cong R(\lambda, -\alpha - \lambda)$ follows from the definition (5.6).

Lemma 5.2. Let $\alpha, \beta \in \mathbb{Z}$.

- (1) *Case $\lambda \geq 2$.* There are three isomorphism classes of $R(\lambda, \alpha)$ given by $\alpha \leq -\lambda$, $\alpha \geq 0$ and $1 - \lambda \leq \alpha \leq -1$ respectively. We denote them by $R(\lambda, <)$, $R(\lambda, >)$ and

$R(\lambda, =)$. We have exact sequences

$$0 \rightarrow M(-\lambda)^- \rightarrow R(\lambda, <) \rightarrow M(\lambda - 2)^\vee \rightarrow 0 \quad (5.8)$$

$$0 \rightarrow M(\lambda - 2)^- \rightarrow R(\lambda, <) \rightarrow M(-\lambda) \rightarrow 0 \quad (5.9)$$

$$0 \rightarrow M(-\lambda) \rightarrow R(\lambda, >) \rightarrow M(\lambda - 2)^{-\vee} \rightarrow 0 \quad (5.10)$$

$$0 \rightarrow M(\lambda - 2) \rightarrow R(\lambda, >) \rightarrow M(-\lambda)^- \rightarrow 0 \quad (5.11)$$

$$0 \rightarrow M(-\lambda) \rightarrow R(\lambda, =) \rightarrow M(\lambda - 2)^- \rightarrow 0 \quad (5.12)$$

$$0 \rightarrow M(-\lambda)^- \rightarrow R(\lambda, =) \rightarrow M(\lambda - 2) \rightarrow 0. \quad (5.13)$$

- (2) *Case* $\lambda \leq 0$. There are three isomorphism classes of $R(\lambda, \alpha)$ given by $\alpha \leq -1$, $\alpha \geq 1 - \lambda$ and $0 \leq \alpha \leq -\lambda$ respectively. We denote them by $R(\lambda, <)$, $R(\lambda, >)$ and $R(\lambda, =)$. Of course we have similar exact sequences as in (1).
- (3) There are two isomorphism classes of $R(1, \alpha)$ given by $\alpha \leq -1$ and $\alpha \geq 0$ respectively. We denote them by $R(1, <)$ and $R(1, >)$. We have exact sequences

$$0 \rightarrow M(-1)^- \rightarrow R(1, <) \rightarrow M(-1) \rightarrow 0 \quad (5.14)$$

$$0 \rightarrow M(-1) \rightarrow R(1, >) \rightarrow M(-1)^- \rightarrow 0. \quad (5.15)$$

- (4) Let $\lambda \neq \mu$. $R(\lambda, \alpha) \cong R(\mu, \beta)$ if and only if $\mu = -\lambda + 2$ and α and β belong to corresponding isomorphism classes in (1) and (2).
- (5) We have $R(\lambda, <)^- \cong R(\lambda, <)^\vee \cong R(\lambda, >)$ and $R(\lambda, =)^- \cong R(\lambda, =)$. $(\cdot)^-$ exchanges (5.8) with (5.10), (5.9) with (5.11), (5.12) with (5.13) and (5.14) with (5.15). $(\cdot)^\vee$ exchanges (5.8) with (5.11), (5.9) with (5.10) and (5.14) with (5.15).

Proof. (1). From (5.7) we deduce $\cdot ef = 0$ and $\cdot fe = 0$ if and only if $n = \alpha$ or $n = \alpha - 1 + \lambda$. Thus, for $\nu \notin \{\lambda, -\lambda + 2\}$ the arrows

$$R(\lambda, \alpha)_\nu \begin{matrix} \xrightarrow{\cdot e} \\ \xleftarrow{\cdot f} \end{matrix} R(\lambda, \alpha)_{\nu-2}$$

are invertible and for $\nu \in \{\lambda, -\lambda + 2\}$ exactly one of the arrows is zero (they cannot both be zero as $R(\lambda, \alpha)$ is not a direct sum). Further, $f = 0$ for $\nu = \lambda$ together with $e = 0$ for $\nu = -\lambda + 2$ cannot occur since $R(\lambda, \alpha)$ is cyclic. We are left with three cases. In each case the allowed range for α follows from the fact that the generator $\bar{1}$ of $R(\lambda, \alpha)$ has weight $-2\alpha - \lambda$.

(2) and (3). See the proof of (1).

(4) $R(\lambda, \alpha) \cong R(\mu, \beta)$ implies that the modules have the same central character. Thus $\mu = -\lambda + 2$. From the definition (5.6) we see $R(\lambda, \alpha) \cong R(-\lambda + 2, \alpha + \lambda - 1)$. The remaining statements follows from (1) and (2).

(5). Remark 5.5 and (1)-(3) imply $R(\lambda, <)^- \cong R(\lambda, >)$ and $R(\lambda, =)^- \cong R(\lambda, =)$. $R(\lambda, <)^\vee \cong R(\lambda, >)$ follows from the proof of (1) and the fact that the definition of $(\cdot)^\vee$ implies $R(\lambda, \alpha)_{\nu-2}f = 0$ if and only if $(R(\lambda, \alpha)^\vee)_\nu e = 0$. \square

The subquotient structure of the $R(\lambda, \alpha)$ is easily deduced from this lemma together with the subquotient structure of the Verma and dual Verma module. E.g. for $\lambda \geq 2$ the simple modules $M(-\lambda), M(-\lambda)^-$ and $L(\lambda-2)$ occur with multiplicity one, while for $\lambda = 1$ the simple modules $M(-1)$ and $M(-1)^-$ occur with multiplicity one.

5.2.3. H^0 and H^1 of $j.\Omega_{\mathbb{C}^\times}$. We find

Theorem 5.3.

$$H^0(j.\Omega_{\mathbb{C}^\times}) = \Omega_{\mathbb{C}^\times}(\mathbb{C}^\times) \cong \begin{cases} R(\lambda, =) & \lambda \leq 0 \\ R(\lambda, =)^\vee & \lambda \geq 2 \\ M(-1) \oplus M(-1)^- & \lambda = 1 \end{cases} \quad H^1(j.\Omega_{\mathbb{C}^\times}) = 0.$$

□

Since $H^1(j_x.\Omega_{\mathbb{C}_x}) = 0$ the long exact sequence of cohomology of (2.10) and (5.5) give the exact sequence

$$0 \rightarrow M(\lambda-2)^\vee \rightarrow H^0(j.\Omega_{\mathbb{C}^\times}) \rightarrow M(-\lambda)^- \rightarrow 0.$$

Applying $(\cdot)^-$ we obtain the exact sequence

$$0 \rightarrow M(\lambda-2)^{-\vee} \rightarrow H^0(j.\Omega_{\mathbb{C}^\times}) \rightarrow M(-\lambda) \rightarrow 0.$$

5.2.4. H^0 and H^1 of $j!\Omega_{\mathbb{C}^\times}$. We find

Theorem 5.4.

$$H^0(j!\Omega_{\mathbb{C}^\times}) \cong \begin{cases} R(\lambda, =) & \lambda \geq 2 \\ M(-\lambda) \oplus M(-\lambda)^- & \lambda \leq 1 \end{cases} \quad H^1(j!\Omega_{\mathbb{C}^\times}) \cong \begin{cases} 0 & \lambda \geq 1 \\ L(-\lambda) & \lambda \leq 0 \end{cases}.$$

□

Since $H^1(\iota_{z*}\mathbb{C}) = 0$ the long exact sequence of cohomology of (2.11) and (5.5) give the exact sequences

$$0 \leftarrow M(\lambda-2) \leftarrow R(\lambda, =) \leftarrow M(-\lambda)^- \leftarrow 0 \quad \lambda \geq 2$$

$$0 \leftarrow M(-\lambda) \leftarrow M(-\lambda) \oplus M(-\lambda)^- \leftarrow M(-\lambda)^- \leftarrow 0 \quad \lambda \leq 1.$$

5.2.5. H^0 and H^1 of $j!_{x \cdot z}\Omega_{\mathbb{C}^\times}$ and $j_{\cdot x!z}\Omega_{\mathbb{C}^\times}$. We find

Theorem 5.5.

$$H^0(j!_{x \cdot z}\Omega_{\mathbb{C}^\times}) \cong R(\lambda, <) \quad H^1(j!_{x \cdot z}\Omega_{\mathbb{C}^\times}) = 0$$

$$H^0(j_{\cdot x!z}\Omega_{\mathbb{C}^\times}) \cong R(\lambda, >) \quad H^1(j_{\cdot x!z}\Omega_{\mathbb{C}^\times}) = 0.$$

□

The statement for $j!_{x \cdot z}$ implies the one for $j_{\cdot x!z}$ because of $j_{\cdot x!z}\Omega_{\mathbb{C}^\times} \cong (j!_{x \cdot z}\Omega_{\mathbb{C}^\times})^-$, Lemma 4.2 and Lemma 5.2(5).

Remark 5.6. The embeddings $\kappa = \iota_{x,z}, j_{x,z}, j : Y \hookrightarrow \mathbb{P}^1$ we consider are all affine. Let \mathcal{M} be a \mathcal{D}_Y -module. Then $H^1(Y, \mathcal{M}) = 0$ implies in a uniform way that the direct image $\kappa_* \mathcal{M} = \iota_{x,z} \mathcal{M}, j_{x,z} \mathcal{M}, j_* \mathcal{M}$ in $\text{mod } \mathcal{D}(\lambda)$ satisfies $H^1(\kappa_* \mathcal{M}) = 0$.

Remark 5.7. The assignment $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(\lambda)$ defines an exact equivalence $\text{mod } \mathcal{D} \rightarrow \text{mod } \mathcal{D}(\lambda)$. Thus, on the level of \mathcal{D} -modules, all twists λ are equivalent. As mentioned in § 2.1 the twist $\lambda = 2$ corresponds to untwisted left \mathcal{D} -modules because $\Omega \cong \mathcal{O}(-2)$. The Beilinson-Bernstein equivalence [BB81] applies in the case $\lambda \geq 2$: Then $H^1 = 0$ and $H^0 : \text{mod } \mathcal{D}(\lambda) \rightarrow \text{mod}(\mathcal{U}\mathfrak{sl}_2)_{\chi_\lambda}$ is an exact equivalence. The vanishing of H^1 in the above examples is consistent with this. The twist $\lambda = 1$ is usually referred to as singular, see (5.5) and Lemma 5.2(3) for examples of how this twist differs from the others. One defines the dual $\mathbb{D}\mathcal{M}$ of a holonomic $\mathcal{D}(\lambda)$ -module \mathcal{M} via the above equivalence $\text{mod } \mathcal{D} \rightarrow \text{mod } \mathcal{D}(\lambda)$. Then \mathbb{D} defines an exact contravariant auto-equivalence of the category of holonomic $\mathcal{D}(\lambda)$ -modules. In the above examples H^0 sends \mathbb{D} to $(\cdot)^\vee$ in the case $\lambda \geq 2$, but we are not aware of a reference proving that such a statement is true in general.

6. OUTLOOK

6.1. Generalization to a semisimple group. Let us describe the setup of a possible generalization of this work from SL_2 to any semisimple algebraic group G over \mathbb{C} . Let B be a Borel subgroup of G , T a maximal torus contained in B and U the unipotent radical of B . Let Φ^+ be the set of positive roots of G w.r.t. B . For an element w of the Weyl group $W = N_G(T)/T$ we denote by \dot{w} any preimage in the normalizer $N_G(T)$ of T . All expressions written below will be independent of such a choice unless mentioned otherwise. Consider the flag variety $X = G/B$ of G . The stratification of X by B -orbits is the well-known Bruhat stratification. It is given by $X = \bigsqcup_{w \in W} X_w$, where $X_w = B\dot{w}B$ is the Bruhat cell associated to w . Each X_w is an affine space whose dimension is given by the length $\ell(w)$ of w .

Fix a simple reflection $s_i \in W$ w.r.t. B . Then $B^{(i)} = B \cap s_i B s_i^{-1}$ defines a closed subgroup of G that is contained in B and contains T . Clearly we have $B^{(i)} = T \ltimes U^{(i)}$, where $U^{(i)}$ is the subgroup of U whose Lie algebra is spanned by the root vectors associated to $\Phi^+ \setminus \{\alpha_i\}$. Let $<$ be the Bruhat-Chevalley order on W . One can show that the $B^{(i)}$ -orbits in X are given by X_w and $s_i X_w$ for $s_i w \not< w$ and $X_w \cap s_i X_w$ for $s_i w < w$. In particular, the stratification of X by $B^{(i)}$ -orbits refines the Bruhat stratification and consists of finitely many locally closed subvarieties of X , which is in contrast with the stratification of X by T -orbits. Note that s_i defines an automorphism of X , depending on the choice of s_i , that interchanges the $B^{(i)}$ -orbits X_w and $s_i X_w$ and leaves the $B^{(i)}$ -orbit $X_w \cap s_i X_w$ invariant. Observe that for $s_i w < w$ the B -orbit X_w decomposes into two $B^{(i)}$ -orbits, namely $X_w = (X_w \cap s_i X_w) \sqcup s_i X_{s_i w}$. In fact $X_w \cap s_i X_w$ is isomorphic as a variety to $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{\ell(w)-1}$. Let us emphasize how we recover for $G = \text{SL}_2$ the setting of the main text: In this case we have $X = \mathbb{P}^1$, our subgroup is the torus, $B^{(i)} = T$, and its orbits

are $\{0\}$, $\{\infty\}$ and \mathbb{C}^\times . For a suitable choice the automorphism s_i of \mathbb{P}^1 coincides with the involution I defined in § 4.1.

Coming back to the general G , given a parameter $\alpha \in \mathbb{C}$ it is clear how to define a local system on the $B^{(i)}$ -orbit $X_w \cap s_i X_w$, generalizing $\Omega_{\mathbb{C}^\times}^{(\alpha)}$ from § 2.5, and the different extensions to X as λ -twisted \mathcal{D} -module. (The twist λ is now an arbitrary integral weight of G w.r.t. T .) It would be interesting to identify, depending on the parameters α and λ , the sheaf cohomology groups as concrete modules over the Lie algebra $\text{Lie } G$ of G thereby generalizing the results of the main text. Here, the word concrete refers to having some algebraic construction of the module, e.g. as a module induced from a proper subalgebra. Concerning the remaining $B^{(i)}$ -orbits, it is known that from the X_w one obtains (the usual highest weight) Verma and dual Verma $\text{Lie } G$ -modules, while the $s_i X_w$ are supposed to give $\text{Lie } G$ -modules differing from the previous ones by a twist by the automorphism of $\text{Lie } G$ defined by s_i .

6.2. Case of the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_2$. Let us finally comment on similar constructions for the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_2$. The discussion is parallel to and depends on § 6.1. Recall that $\widehat{\mathfrak{sl}}_2$ is constructed from a triple $(\mathfrak{h}, (\alpha_i)_{i \in \{0,1\}}, (\alpha_i^\vee)_{i \in \{0,1\}})$, where \mathfrak{h} is a three dimensional \mathbb{C} -vector space and the generalized Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. It has a triangular decomposition $\widehat{\mathfrak{sl}}_2 = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and elements $f_i \in \mathfrak{n}^-$, $e_i \in \mathfrak{n}$, $i \in \{0,1\}$. The flag variety X of $\widehat{\mathfrak{sl}}_2$ has well-known formulations either as an ind-projective ind-variety [BD91] or as a scheme [Kas90] over \mathbb{C} . In the following we work with the latter version. The Weyl group W of $\widehat{\mathfrak{sl}}_2$ is generated by s_0, s_1 with relations $s_0^2 = s_1^2 = 1$. In [KT95] it is explained how to attach to $w \in W$ a (finite dimensional) Bruhat cell $X_w \cong \mathbb{A}^{\ell(w)}$ that is locally closed in X . One has a replacement for s_1 and for $s_1 w < w$ the intersection $X_w \cap s_1 X_w$ is still defined and isomorphic to $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{\ell(w)-1}$. Due to [KT95] one disposes of a category of λ -twisted right \mathcal{D} -modules on X with support in the closure $\overline{X_w}$ of X_w in X . To the objects of this category [KT95] attach cohomology groups and construct a $\widehat{\mathfrak{sl}}_2$ -action on these. Consequently it makes sense to try to identify the cohomology groups of the \mathcal{D} -module extensions of the local system on $X_w \cap s_1 X_w$ analogous to $\Omega_{\mathbb{C}^\times}^{(\alpha)}$ as concrete $\widehat{\mathfrak{sl}}_2$ -modules, completely parallel to what was proposed in § 6.1. In fact, the corresponding identification of the cohomology groups for X_w instead of $X_w \cap s_1 X_w$ and antidominant λ constitutes the main theorem [KT95][Theorem 3.4.1] of [KT95]. Let us consider the simplest case $w = s_1$, then $X_w \cap s_1 X_w \cong \mathbb{A}^1 \setminus \{0\}$. We can then prove that for $-\lambda(\alpha_1^\vee) \geq 2$ the global sections of the $!$ - resp. $*$ -extension form an induced module of the form $\mathcal{U} \widehat{\mathfrak{sl}}_2 \otimes_{\mathcal{U} \mathfrak{p}_1} (\mathbb{C}_\lambda \otimes_{\mathbb{C}} M)$ with $M = R(-\lambda(\alpha_1^\vee), \alpha)$ as defined in (5.6) resp. $M = R(-\lambda(\alpha_1^\vee), \alpha)^\vee$. Here the subalgebra $\mathfrak{p}_1 = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathbb{C} f_1$ acts on $\mathbb{C}_\lambda \otimes_{\mathbb{C}} M$ via the map of Lie algebras

$$\mathfrak{p}_1 \twoheadrightarrow \{h \in \mathfrak{h} \mid \alpha_1(h) = 0\} \oplus (\mathbb{C} f_1 \oplus \mathbb{C} \alpha_1^\vee \oplus \mathbb{C} e_1) .$$

Note that this induced module is neither a highest nor a lowest weight module. Such $\widehat{\mathfrak{sl}}_2$ -modules were introduced and analyzed in [SS97] and [FST98] under the name relaxed Verma modules. It is natural to ask whether a similar result holds for general w and we hope to address this question in a future publication.

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DEPARTEMENT MATHEMATIK, ETH ZUERICH, 8092 ZUERICH, SWITZERLAND

E-mail address: `claudio.eicher@math.ethz.ch`